I. INTRODUCTION

In this paper I will develop an Herbrand theory for higher order logic. In a previous paper (1) I gave Skolem and Herbrand theorems for first order logic. The theory was a further development of the completeness theorems of Beth, Hintikka and Smullyan. Now Prawitz (2) and Takahashi (3) have given similar completeness-theorems for higher order logic. The original idea for this paper was to use the result of Prawitz and Takahashi and then see how far I could go along the lines of my previous paper.

In first order logic one first eliminates the general quantifiers by introducing functionsymbols for them. Then the restricted quantifiers are eliminated by introducing large enough finite $\bigwedge$ for $\forall$. This procedure breaks down in some obvious places in higher order logic. I will show in this paper that when the procedure is amended in a reasonable way it works.

The plan here is as follows:

i) I assume the reader knows my previous paper (1). In fact where there is almost a parallel development to first order logic I will just indicate the differences. This I hope will both shorten the paper and increase its perspicuity.

ii) The logical system is given as a sequential calculus for classical higher order logic without extensionality.

iii) The first new problem is to get control over the quantifiers we "get" from a sequent in a proof of it. In first order logic the quantifiers are just the quantifiers appearing in the sequent. (By the subformula property.) In higher order logic this does not work. One may get new quantifiers in the following two processes

$$\forall X F[X] \longmapsto F[S] \quad \text{and} \quad T \in \lambda Y G[Y] \longmapsto G[T].$$

I introduce "the potential Herbrandprocesses" and from them "indices". With them I get names for all quantifiers which may occur in a proof of a sequent.

iv) With each index for a general quantifier I introduce a Skolemfunction for the quantifier. Say we have a general quantifier $\forall X$ and it is bound by the restricted quantifiers $\forall Y_1, \dots, \forall Y_N$. Now it may happen that we must analyze some of the restricted quantifiers in a particular way to get $\forall X$. Those quantifiers give rise to the singular arguments of the Skolemfunction for $\forall X$. The other quantifiers give the regular arguments.

v) With the potential Herbrandprocesses and the Skolemfunctions I define the Herbrandprocesses.

vi) I extend the notions to trees of sequents, not only a single sequent.

vii) The Herbrandtheorem is formulated and proved. As in first order logic I work with trees of sequents.

II: THE LOGICAL SYSTEM LHK

The logical system is a sequential formulation of classical higher order logic without extensionality.

Language

Connectives : M (finite conjunction), $\neg$

Quantifier $\forall$

Types $0,1,2,\ldots,n,\ldots$

Parameters of each type n A_1^n, A_2^n, $\ldots, B^n, C^n, \ldots$

Variables of each type n X_1^n, $X_2^n, \ldots Y^n, Z^n, \ldots$

For each type n a constant E^n

Predicatesymbols for types $n_1,\ldots,n_m$

Abstraction operator λ

Elementrelation ϵ

Functionsymbols. We do not include functionsymbols yet, but could have done it without too much extra work. Later when we come to the Skolemfunctions we need functionsymbols. In similar way as below we define quasiformulae, formulae, quasiterms, and terms in the extended language. There is one unusual point about the functionsymbols, we have two forms of argumentplaces - singular and regular. In the regular argumentplaces we can use any term as argument. In the singular argumentplaces we are only allowed to use terms which has not a λ as outermost symbols. We will come back to the functionsymbols in the section on Skolemfunctions.

Formulae and terms are defined by:

1. Atomic quasiformulae are either

i) $P(T_1,\ldots,T_N)$ with P predicatesymbol and $T_1,\ldots,T_N$ quasiterms of appropriate types ; or

ii) $T_1 \epsilon T_2$ with T_1, T_2 quasiterms of consecutive types.

2) Quasiformulae are built up as usual by $M, \neg \forall$.

3) Formulae are quasiformulae without free variables.

4) Quasiterms are either

i) parameters;

ii) variables;

iii) constants; or

iv) $\lambda X F[X]$ with X variable in quasiformula $F[X]$ and the quasiterm is of type 1+type of X.

5) Terms are quasiterms without free variables.

Equality. Two quasiformulae, formulae, quasiterms, or terms are equal if they are equal up to the names of their variables.

Sequents are defined as a pair of two finite sequences of formulae. We have quasisequents if we have only quasiformulae.

Subformula. A subformula of a quasiformula F is a quasi-formula which is a part of F. A proper subformula of a quasiformula F is a subformula of F which is not a subformula of an atomic subformula of F, i,e. it is not part of a term of the form $\lambda X G[X]$. Similarily we define subformula and proper subformula of a sequent.

In some cases below we will also have two notions - one ordinary and one proper. The proper will arise from treating formulae like $T \epsilon \lambda X G[X]$ as not composite.

Reducuble and irreducible. A reducible quasiformula is a quasiformula which contains a proper subformula of the form $T \epsilon \lambda X G[X]$. An irreducible quasiformula is a quasiformula which is not reducible. An irreducible sequent is a sequent which does not contain a reducible formula.

AXIOMS OF LHK:

$$\Gamma_1, A, \Gamma_2 \longrightarrow \Delta_1, A, \Delta_2 \qquad \text{A irreducible atomic formula.}$$

RULES OF LHK: As in LK with addition of the following two:

$$\lambda \longrightarrow \qquad \frac{\Gamma, F\{T\} \longrightarrow \Delta}{\Gamma, T \in \lambda X F\{X\} \longrightarrow \Delta}$$

$$\longrightarrow \lambda \qquad \frac{\Gamma \longrightarrow F\{T\}, \Delta}{\Gamma \longrightarrow T \in \lambda X F\{X\}, \Delta}$$

Having defined the logical system LHK we carry over some of the theory of LK.

Precedes. Consider the notion of 'Immediately precedes' in LK. In LHK we have two notions 'immediately precedes' and 'immediately properly precedes'. To each part of a premiss in a rule we associate in a natural way a part of the conclusion. We say that the one part immediately precedes the other. In the same way we get 'immediately properly precedes' except that parts of principal formulae of $\longrightarrow \lambda$ or $\lambda \longrightarrow$ have no proper predecessors, i.e. parts of $F\{T\}$ in $\longrightarrow \lambda$ and $\lambda \longrightarrow$ above have no proper predecessor. — As in LK we define then 'immediately (properly) succeeds', '(properly) precedes', '(properly) succeeds', and 'in the same (Proper) strand!

Analysis. The 'terms' we introduce in our trees for $\forall \longrightarrow$ consists of a quasiterm and a substitution of parameters and constants for the free variables. 'The analysis of a formula' is defined as in LK except that we have quasiterms and substitutions for the free variables.

Positive and negative. Positive and negative occurrences are defined as in LK. Note that we then define positive and negative occurrences only for proper subformulae.

General and restricted quantifiers are defined as in Lk. So only quantifiers occurring outermost in proper subformulae are general or restricted. We call such quantifiers proper.

Trees. A tree over a sequent $\Gamma \longrightarrow \Delta$ is a tree of sequents with $\Gamma \longrightarrow \Delta$ at the downmost node and such that

i) a sequent at a node and the sequents at its successornodes are related as conclusion and premisses of one of the rules of LHK; and

ii) At $\forall \longrightarrow$ we introduce quasiterms built up from symbols in $\Gamma \longrightarrow \Delta$ and substitutions of the free variables in the quasiterms by constants or parameters introduced by $\longrightarrow \forall$ somewhere in the tree.

Later we will introduce LHK-trees (as generalization of classical trees), but for that we need a notion of 'two occurrences of proper quantifiers beeing occurrences of the same quantifier! In LK this is done by saying that the two quantifier occurrences shall be in the same strand, but this does not work in LHK. The correct notion in LHK (as we shall see) is that the two quantifier-occurrences have the same index.

III. INDICES.

We come now to the first real departure from Lk. In LK we can trace every general quantifier in a classical tree down to a general quantifier in the bottomsequent. The same with restricted quantifiers. It is not so in general in LHK. For example consider the tree:

$$\underline{\forall X\ (X \in E)\ \xrightarrow{6}\ \forall X\ (X \in E)}$$

$$\underline{E \in \lambda Y\ (\forall X (X \in Y)) \xrightarrow{5} E \in \lambda Y\ (\forall X (X \in Y))}$$

$$\underline{E \in \lambda Y\ (\forall X (X \in Y)) \wedge \neg E \in \lambda Y\ (\forall X (X \in Y))\ \xrightarrow{4}}$$

$$\underline{\lambda Y\ (\forall X \cdot (X \in Y)) \in \lambda Z\ (E \in Z \wedge \neg E \in Z) \xrightarrow{3}}$$

$$\underline{\forall U\ (U \in \lambda Z\ (E \in Z \wedge \neg E \in Z))\ \xrightarrow{2}}$$

$$\underline{\forall V \forall U\ (U \in V)\ \xrightarrow{1}}$$

We have omitted some nodes in the tree. — In 6 there are two occurrences of $\forall X$ — one general and one restricted. They can both be traced down to a single occurrence in 3. We will get into difficulties if we try to let the $\forall X$ in 3 be either general or restricted. The $\forall X$ in 3 can then be traced down to U in 2 and 1. The $\forall X$ which will appear higher up in the tree is hidden in 2 and 1.

The potential Herbrand processes in this section is a method to uncover hidden quantifiers in a sequent. We have two basic operations RED and $H_{\pi,\mathfrak{D}}$ (for various π and $\mathfrak{D}$) which we introduce below.

DEFINITION $\pi, \pi_1, \ldots, \pi_N$ below denote positions.

1. RED_π is the operation on sequents which to a quasi-sequent $\Gamma \longrightarrow \Delta$ gives

   i) if at position π in $\Gamma \longrightarrow \Delta$ there occurs a proper subformula of form $T \in \lambda X F[X]$, then we get RED_π $(\Gamma \longrightarrow \Delta)$ from $\Gamma \longrightarrow \Delta$ by putting $F[T]$ for the $T \in \lambda X F[X]$ at position π. We assume that we rename variables so as to avoid conflicts between them ;

   ii) else RED_π $(\Gamma \longrightarrow \Delta\ ) = \Gamma \longrightarrow \Delta$

2. Similarly we define $RED_{\pi_1,\ldots,\pi_N}$ where we get rid of reducible subformulae at positions $\pi_1,\ldots,\pi_N$.
3. RED is the operation we get by applying RED_π for various π until we get an irreducible quasisequent. RED is easily seen to be well-defined up to the names of the variables.

DEFINITION Let π be a position and $\mathfrak{D}$ a finite sequence of quasiterms of the same type n. $H_{\pi,\mathfrak{D}}$ is the operation which to a quasisequent $\Gamma \longrightarrow \Delta$ gives:

i) if at position π in $\Gamma \longrightarrow \Delta$ there occurs a restricted quantifier $\forall X$ with X of type n, we get $H_{\pi,\mathfrak{D}}(\Gamma \longrightarrow \Delta)$ from $\Gamma \longrightarrow \Delta$ by replacing $\forall X$ with $\bigwedge_{X\in\mathfrak{D}}$. We assume that we rename variables so as to avoid conflicts between them;

ii) else $H_{\pi,\mathfrak{D}}(\Gamma \longrightarrow \Delta) = \Gamma \longrightarrow \Delta$.

DEFINITION A potential Herbrand process over $\Gamma \longrightarrow \Delta$ is an operation denoted by $(\pi_1,\mathfrak{D}_N) \ldots . (\pi_N,\mathfrak{D}_N)$. It acts on $\Gamma \longrightarrow \Delta$ as $RED \circ H_{\pi_N,\mathfrak{D}_N} \circ RED \circ \ldots RED \circ H_{\pi_1,\mathfrak{D}_1} \circ RED$. Here $\pi_1,\ldots,\pi_N$ are positions, $\mathfrak{D}_1,\ldots \mathfrak{D}_N$ are finite sequences of quasiterms built up from symbols in $\Gamma \longrightarrow \Delta$.

<u>Tracing</u> To each part in $RED_\pi(\Gamma \longrightarrow \Delta)$ we associate in a natural way a part in $\Gamma \longrightarrow \Delta$. We say that the part in $RED_\pi(\Gamma \longrightarrow \Delta)$ is traced back to the part in $\Gamma \longrightarrow \Delta$. Similarly to RED_π for the operations RED and $H_{\pi,\mathfrak{D}}$ and for all potential Herbrand processes.

Example:

Let $\Gamma \longrightarrow \Delta$ be $A \longrightarrow B \wedge (\lambda XF[X] \in \lambda YG\ Y)$ and $RED_\pi(\Gamma \longrightarrow \Delta)$

$A \longrightarrow B \wedge G[\lambda XF[X]]$.

Then A and B in $RED_\pi(\Gamma \longrightarrow \Delta)$ are traced back to A and B in $\Gamma \longrightarrow \Delta$. $\lambda XF[X]$ in RED_π $(\Gamma \longrightarrow \Delta)$ is traced back to

$\lambda XF[X]$ in $\Gamma \longrightarrow \Delta$. A subformula of $G[\lambda XF[X]]$ in $RED_\pi(\Gamma \longrightarrow \Delta)$ involving $\lambda XF[X]$ is traced back to $\lambda XF[X]$ and a part of $G[Y]$.

The potential Herbrand processes are the means by which we uncover proper quantifiers which may occur in trees. It is clear that any proper quantifier which may occur in a tree over $\Gamma \longrightarrow \Delta$ can be got as a proper quantifier by a potential Herbrand process over $\Gamma \longrightarrow \Delta$. This will be made precise by the indices below and the assignment of indices to formulae in trees over $\Gamma \longrightarrow \Delta$

DEFINITION An elementary Herbrand process over $\Gamma \longrightarrow \Delta$ is a potential Herbrand process over $\Gamma \longrightarrow \Delta$, $(\pi_1,T_1)\ldots.(\pi_N,T_N)$, where the domains consist of single quasiterms $T_1,\ldots T_n$. Besides we have

i) In $\Gamma_1 \longrightarrow \Delta_1 = RED\ (\Gamma \longrightarrow \Delta)$ there occurs a restricted quantifier $\forall X_1$ at position π_1 and X_1 and T_1 are of the same type.

ii) For $k = 2,3,\ldots,N$: In $\Gamma_k \longrightarrow \Delta_k = RED\ H_{\pi_{k-1},T_{k-1}}(\Gamma_{k-1} \longrightarrow \Delta_{k-1})$ there occurs a restricted quantifier $\forall X_k$ at position π_k of the same type as T_k and it can be traced back to $\forall X_{k-1}$ at position π_{k-1} in $\Gamma_{k-1} \longrightarrow \Delta_{k-1}$.

DEFINITION Assume we have an elementary Herbrand process over $\Gamma \longrightarrow \Delta$, $(\pi_1,T_1)\ldots(\pi_N,T_N)$, as above. An index over $\Gamma \longrightarrow \Delta$ is a pair $\langle(\pi_1,T_1)\ldots(\pi_N,T_N),\pi\rangle$ where π is a position which in $[(\pi_1,T_1)\ldots(\pi_N,T_N)]\ (\Gamma \longrightarrow \Delta)$ can be traced back to $\forall X_N$ at position π_N in $\Gamma_N \longrightarrow \Delta_N$ ($\forall X_N,\pi_N,\Gamma_N \longrightarrow \Delta_N$ as defined above). As indices we have also all $\langle\ ,\pi\rangle$ where π is a position in RED $(\Gamma \longrightarrow \Delta)$.

We regard the indices of $\Gamma \longrightarrow \Delta$ as generalizations of positions in $\Gamma \longrightarrow \Delta$. In first order logic we can work with positions. Every quantifier analyzed in a classical tree can be traced down to a position in the bottomsequent. We do not get more quantifiers in a tree in LK than appears in the bottom-sequent. In LHK the situation is quite different as we have seen. Instead of tracing down the formulae in trees in LHK to the bottomsequent we assign indices. Below we write down in some detail the assignment of indices to proper formulae in trees and proper subformulae in potential Herbrand processes.

Assignment of indices in potential Herbrand processes.

Given a potential Herbrand process P over $\Gamma \longrightarrow \Delta$. Then to each proper subformula occurring in the process we assign an index as follows:

Say P is $(\pi_1, \mathfrak{D}_1) \ldots . (\pi_N, \mathfrak{D}_N)$.

We put $P_i = (\pi_1, \mathfrak{D}_1) \ldots . (\pi_i, \mathfrak{D}_i) \quad 1 \leq i \leq N$

i) Each proper subformula in RED $(\Gamma \longrightarrow \Delta)$ is assigned index $< , \pi >$ where π is the position it occurs in.

ii) Say we have assigned indices to the proper subformula in RED $(\Gamma \longrightarrow \Delta)$, $P_1(\Gamma \longrightarrow \Delta), \ldots , P_i(\Gamma \longrightarrow \Delta)$. Let F be a proper subformula of $P_{i+1}(\Gamma \longrightarrow \Delta)$.

a) If F can be traced back to a proper subformula in $P_i(\Gamma \longrightarrow \Delta)$, then F is assigned its index.

b) Else at position π_{i+1} in $P_i(\Gamma \longrightarrow \Delta)$ there occurs a restricted variable and F is traced back to it. Then in $H_{\pi_{i+1} \mathfrak{D}_{i+1}} \circ P_i(\Gamma \longrightarrow \Delta)$ F is traced back to one of the quasi-terms in $\mathfrak{D}_{i+1}$, say T. Let the proper subformula occurring at π_{i+1} in $P_i(\Gamma \longrightarrow \Delta)$ have assigned index $<Q,q>$ F occurs in $[Q(q,T)](\Gamma \longrightarrow \Delta)$ at some position p.

Then F in $P_{i+1}(\Gamma \longrightarrow \Delta)$ is assigned index $\langle Q(q,T),p.\rangle$

<u>Assignment of indices in trees.</u> Given a sequent $\Gamma \longrightarrow \Delta$ and a tree $\mathcal{T}$ over it. By induction we assign indices to proper subformulae occurring in $\mathcal{T}$.

i) Each proper subformula occurring in the bottomsequent of $\mathcal{T}$, $\Gamma \longrightarrow \Delta$, has already assigned an index.

ii) Now consider a proper subformula F of a sequent $\Gamma_1 \longrightarrow \Delta_1$ at a node Y in $\mathcal{T}$ and assume we have already assigned indices to all proper subformulae at nodes below Y.

There are two cases to consider:

a) There are no variable X which precedes F in $\mathcal{T}$.
Then F corresponds to a proper subformula of $RED\ (\Gamma \longrightarrow \Delta)$ and is assigned its index.

b) Else there is a variable which precedes F in $\mathcal{T}$.
Let the uppermost occurrence of such a variable which precedes F be as $\forall XG[X]$. The $\forall X$ must be a restricted variable since F is proper. Say $\forall XG[X]$ occurs as $\Gamma_2, \forall XG[X] \longrightarrow \Delta_2$ and above we have $\Gamma_2, G[T] \longrightarrow \Delta_2$ with T preceding F. $\forall XG[X]$ has assigned index (P,p). Then $P(p,T)$ is an elementary Herbrand process and F corresponds to a proper subformula $[P(p,T)](\Gamma \longrightarrow \Delta)$ at some position q. We assigne index $\langle P(p,T),q\rangle$ to F.

We are now ready to proceed with the development of the theory for LHK which is parallel to the one of LK.

<u>LHK - tree</u> An LHK - tree over $\Gamma \longrightarrow \Delta$ is a tree over $\Gamma \longrightarrow \Delta$ where :

i) parameters introduced by $\longrightarrow \forall$ are distinct if we analyze quantifiers not with the same index or with distinct analysis; and

ii) there is a well-order of the parameters (called less than or equal) such that for any parameter a introduced at a node ν by $\longrightarrow \forall$, all parameters occurring at nodes below ν are strictly less than a.

<u>Secured</u> nodes, branches, trees as in LK.

LEMMA Given a secured LHK-tree over $\Gamma \longrightarrow \Delta$. We can find a finite secured LHK-tree over $\Gamma \longrightarrow \Delta$.

PROVABILITY LEMMA If we have a secured LHK-tree over $\Gamma \longrightarrow \Delta$, then $\vdash_{LHK} \Gamma \longrightarrow \Delta$.

<u>Analyzing</u> branch (and tree) is defined as in LK except that we change v slightly to v' and add condition viii:

v') if $\forall X F[X]$ occurs in an antecedent in β, then for every quasiterm of the same type as X and for every substitution of constants and parameters introduced by $\longrightarrow \forall$ we have for the resulting term T that $F[T]$ occurs as a successor to a formula in the same strand of formulae as $\forall X F[X]$ in an antecedent in β;

vii) if $T \epsilon \lambda X F[X]$ occurs in β as a formula), then $F[T]$ occurs as a successor to $T \epsilon \lambda X F[X]$ in β.

ANALYZING LEMMA To any sequent we can find an analyzing LHK-tree over it.

FALSIFIABILITY LEMMA If we have a not-secured analyzing branch in an LHK-tree over $\Gamma \longrightarrow \Delta$, then we can find a falsifying model for $\Gamma \longrightarrow \Delta$.

The falsifiability lemma is exactly what Takahashi [3] and Prawitz [2] proves. Here is the only place in this paper where we use their work.

SOUNDNESS LEMMA For any sequent $\Gamma \longrightarrow \Delta$, if $\vdash_{LHK} \Gamma \longrightarrow \Delta$, Then there are no falsifying models for $\Gamma \longrightarrow \Delta$.

COMPLETENESS THEOREM For any sequent $\Gamma \longrightarrow \Delta$, $\vdash_{LHK} \Gamma \longrightarrow \Delta$ if and only if there are no falsifying models for $\Gamma \longrightarrow \Delta$.

CONSISTENCY THEOREM For any sequent $\Gamma \longrightarrow \Delta$, we have exactly one of i and ii below:

i) a secured LHK-tree over $\Gamma \longrightarrow \Delta$;

ii) an LHK-tree over $\Gamma > \Delta$ with not-secured analyzing branch.

Strong LHK-tree An LHK-tree is strong if: parameters introduced by $\longrightarrow \forall$ are distinct if and only if we analyze quantifier with the same index and with the same analysis.

STRONG ANALYZING LEMMA To any sequent we can find a strong, analyzing LHK-tree over it.

The proofs of the theorems and lemmata above can be carried over from [1] with only the obvious changes.

LHK-morphisms, provability morphisms, analyzing morphisms, falsifiability morphisms, LHK-isomorphism as in LK.

IV. SKOLEM FUNCTIONS.

We regard the indices of $\Gamma \rightarrow \Delta$ as generalizations of positions. To the positions in LK we assign the restricted variables binding it. Similarly in LHK, but here we have two ways of binding – the ordinary binding and the free variables in the index.

DEFINITION Given an index $\langle P,p \rangle$ of $\Gamma \longrightarrow \Delta$. The regular variables binding $\langle P,p \rangle$ are the restricted variables binding position p in $P(\Gamma \longrightarrow \Delta)$. The singular variables binding $\langle P,p \rangle$ are the free variables in the quasiterms in P.

In LK we assign to every general variable in a sequent $\Gamma \longrightarrow \Delta$, a functionsymbol with an argumentplace for every restricted variable binding it.

Skolemfunctions To every index i of $\Gamma \longrightarrow \Delta$ which is an index of a general variable we introduce the Skolemfunction of the index f_i. If a functionsymbol with singular argumentplaces for each singular variable binding it and regular argumentplaces for each regular variable binding it. Each argumentplace is of the same type as the variable. The result of applying F_i is a term of same type as the general variable at i.

See beginning of section II for 'singular and regular argumentplaces' of functionsymbols.

DEFINITION. An actual Herbrandprocess over $\Gamma \longrightarrow \Delta$ is an operation on $\Gamma \longrightarrow \Delta$ given by:

i) a potential Herbrand process of $\Gamma \longrightarrow \Delta$, and

ii) substitutions of terms, not starting with a λ from the language extended with Skolemfunctions of $\Gamma \longrightarrow \Delta$, for all the free variables in the quasiterms of the potential Herbrand process.

DEFINITION An Herbrand process over $\Gamma \longrightarrow \Delta$ is an actual Herbrand process which gives applying it to $\Gamma \longrightarrow \Delta$ a sequent without restricted variables.

We take a look at our definition of actual Herbrand processes. Our reason for excluding terms with a λ as outermost symbol is that those terms can and are the only ones that can generate new proper quantifiers. We want to split up each Herbrand-process over $\Gamma \longrightarrow \Delta$ into two – a potential Herbrand-process over $\Gamma \longrightarrow \Delta$ in the language of $\Gamma \longrightarrow \Delta$, and a substitution of terms from the extended language. The split up shall be such that any proper quantifier generated by the Herbrand-process is already generated by the potential Herbrand process.

We assigned indices to each proper subformula occurring in a potential Herbrand process. The same can of course be done to every proper subformula occurring in an actual Herbrand-process. Now there is more information we want about the proper subformula.

<u>Assignment of places.</u> A place is a triple $\langle i,a,b \rangle$ - i is an index, a is called the singular analysis, and b is called the regular analysis. Let F be a proper subformula occurring in an actual Herbrand-process $\mathfrak{a}$ over $\Gamma \longrightarrow \Delta$. F is assigned place $\langle i,a,b \rangle$ where:

i) i is the index assigned to F

ii) a is given by the substitution of terms for the singular variables binding i

iii) Some of the regular variables binding i have been analyzed in the occurrence F. b gives the analysis of those regular variables.

From the above it is clear that a gives terms not with a λ as outermost symbol and in the extended language. b gives terms in the extended language. The set of places assigned to proper subformulae occurring in $\mathfrak{a}$ is called the places of $\mathfrak{a}$.

DEFINITION Given $\mathcal{A}_1,\mathcal{A}_2$ actual Herbrand processes over $\Gamma \longrightarrow \Delta$. Then $\mathcal{A}_1 < \mathcal{A}_2$ if and only if the places of $\mathcal{A}_1 \subseteq$ the places of $\mathcal{A}_2$.

We have assigned places to actual Herbrand processes. Conversely we will show how to certain finite sets of places we can find an actual Herbrand-process which the set is the places of.

We work with the sequent $\Gamma \longrightarrow \Delta$. Let $<<E,p>,a,b>$ be a place in an actual Herbrandprocess over $\Gamma \longrightarrow \Delta$. To this place there is a canonical actual Herbrand-prosess given as follows:

i) Observe that every variable analyzed in the regular analysis b occurs in $E(\Gamma \longrightarrow \Delta)$, say at positions $\pi_1,\ldots\pi_N$. We can then extend the potential Herbrand process E by $(\pi_1 T_1)\ldots.(\pi_N T_N)$ where $T_1,\ldots T_N$ are the quasiterms given by b.

ii) The potential Herbrand process is then given by E and $(\pi_1 T_1)\ldots(\pi_N T_N)$. The substitution of terms for the free variables in E is given by the singular analysis a, and for the free variables in $T_1,\ldots T_N$ by the regular analysis b. We call this actual Herbrand-process the <u>canonical</u> actual Herbrand process associated with the place. A set of places of a sequent $\Gamma \longrightarrow \Delta$ is <u>closed</u> if to every place p it contains the places of the canonical actual Herbrand process associated with p.

THEOREM $\Gamma \longrightarrow \Delta$ is a sequent. Let P be a finite, non-empty, closed set of places of $\Gamma \longrightarrow \Delta$. Then there is an actual Herbrand process $\mathcal{A}$ such that P is the places of $\mathcal{A}$.

Proof:

Since P is non-empty and closed it must contain the places of the empty actual Herbrand-process. (i.e. the Herbrandprocess

(π) where π is a position not in $\Gamma \longrightarrow \Delta$). Since P is finite and contains the places of an actual Herbrand process we can find an actual Herbrand process $\mathfrak{a}$ with the places of $\mathfrak{a}$ $P_{\mathfrak{a}} \subseteq P$ and $P_{\mathfrak{a}}$ maximal with respect to $\subseteq$.

We will show that $P_{\mathfrak{a}} = P$

Assume not. Let $\langle E(pT), r\rangle$ be an index of minimal length of a place $\langle\langle E(pT), r\rangle, a, b\rangle \in P - P_a$. There is then a place of $\Gamma \longrightarrow \Delta$: $\langle\langle E, p\rangle, a', b'\rangle$ where $a' \subseteq a$ and $b' \subseteq b$. Since P is closed we must have $\langle\langle E, p\rangle, a', b'\rangle \in P$. By minimality $\langle\langle E, p\rangle, a', b'\rangle \in P_a$.
$\langle\langle E, p\rangle, a', b'\rangle$ is a place of a restricted quantifier of the same type as T.
The quantifier is either not analyzed in $\mathfrak{a}$ or it is not analyzed with the term T (with free variables from a substituted.
In both cases it is straight forward how to find an actual Herbrand process $\mathfrak{a}^*$ with $P_{a^*} \subsetneq P_a$ and P_a $\{\langle\langle E(pT), r\rangle, a, b\rangle\} \subseteq P_{a^*}$
We get a contradiction and conclude $P_a \neq P$.

THEOREM Let P be a finite set of places of $\Gamma \longrightarrow \Delta$. There is a finite, non-empty, closed set of places of $\Gamma \longrightarrow \Delta$, P^* with $P \subseteq P^*$.

Proof:

Obvious.

THEOREM To any actual Herbrand process over $\Gamma \longrightarrow \Delta$ we can find an Herbrand process $\mathfrak{a}^*$ over $\Gamma \longrightarrow \Delta$ with $\mathfrak{a} < \mathfrak{a}^*$.

Proof:

After applying $\mathfrak{a}$ on $\Gamma \longrightarrow \Delta$ we get a sequent $\mathfrak{a}(\Gamma \longrightarrow \Delta)$ which may contain restricted variables. Take $\mathfrak{a}^*$ as the Herbrand process we get by first applying $\mathfrak{a}$ and then analyze the restricted variables in $\mathfrak{a}(\Gamma \longrightarrow \Delta)$ with constants of appropriate types. It is clear that $\mathcal{A}^*(\Gamma \longrightarrow \Delta)$ does not contain any restricted variables.

THEOREM To any finite set P of places of $\Gamma \longrightarrow \Delta$, we can find an Herbrand process $\mathcal{A}$ over $\Gamma \longrightarrow \Delta$ with $P \subseteq$ the places of $\mathcal{A}$.

Proof:

From the theorems above.

V. THE HERBRAND THEOREM.

In the preceding section we only worked with sequents. We will now show how to extend the notions to LHK-trees. To fix our ideas we will work with an LHK-tree $\mathcal{T}$ over $\Gamma \longrightarrow \Delta$ and let $\mathcal{A}$ be the set of new parameters introduced in $\mathcal{T}$ by $\longrightarrow \forall$

Assignment of places in $\mathcal{T}$ using $\mathcal{A}$.

We have previously shown how to associate with each formula in $\mathcal{T}$ an index. Now we have also associated with each formula in $\mathcal{T}$ its analysis where we use symbols from $\mathcal{A}$. It is then clear how to assign a place to every formula in $\mathcal{T}$ using the symbols from $\mathcal{A}$.

Our task now is to use the Skolemfunctions of $\Gamma \longrightarrow \Delta$ to eliminate the symbols of $\mathcal{A}$. Let $\mathcal{S}$ be the set of all terms $f(\vec{s},\vec{r})$ where f is a Skolemfunction of $\Gamma \longrightarrow \Delta$, $\vec{s}$ and $\vec{r}$ are terms built up from symbols in $\Gamma \longrightarrow \Delta$, constants, and Skolemfunctions of $\Gamma \longrightarrow \Delta$ ($\vec{s}$ - singular arguments and $\vec{r}$ - regular arguments.) Below assume $\mathcal{A} \neq \emptyset$. Else it is trivial.

The natural map from $\mathcal{A}$ into $\mathcal{S}$.

The map is defined by induction on the well-ordering of $\mathcal{A}$ given by $\mathcal{T}$. (See definition of LHK-tree.) Assume all parameters strictly less than a have defined their image in $\mathcal{S}$.

a may be introduced by $\longrightarrow \forall$ a number of places in $\mathcal{T}$ but by definition of LHK-tree all places where it is introduced it is from formulae with the same analysis and the same index. The index is an index of a general variable and there is hence a Skolemfunction f of that index. Say the singular analysis is $\vec{s}$ and the regular analysis is $\vec{r}$. (using symbols from $\mathcal{A}$) of the place we introduce a. Now in $\vec{s}$ and in $\vec{r}$ there occurs only parameters from $\mathfrak{A}$ strictly less than a (by definition of LHK-tree). They have an image in $\mathfrak{S}$. Let $\vec{s}'$ and $\vec{r}'$ be the terms we get by substituting the image in $\mathfrak{S}$ of a parameter for the parameter. Observe that since $\vec{s}$ consists of terms not starting with a λ, the same is also true for $\vec{s}'$. Hence $f(\vec{s}',\vec{r}')$ is defined and we let it be the image of a.

Properties of the natural map $\mathfrak{A} \longrightarrow \mathfrak{S}$.

1. If $\mathcal{T}$ is strong the map is injective.
2. If $\mathcal{T}$ is analyzing with at least one $\longrightarrow$ the map is surjective.

Both properties follows easily from the definition of strong and analyzing.

Assignment of places in $\mathcal{T}$ using Skolemfunctions.

Having the natural map $\mathfrak{A} \longrightarrow \mathfrak{S}$ we eliminate the symbols of $\mathfrak{A}$ in the places of $\mathcal{T}$ by instead using their image in $\mathfrak{S}$. Observe that this assignemt of places matches with the natural map so that if we have

$$\frac{\Gamma_1 \longrightarrow F_a,\Delta_1}{\Gamma_1 \longrightarrow \forall x\, Fx,\Delta_1}$$

somewhere in $\mathcal{T}$ and $\forall xFx$ has place $\langle i,A,B\rangle$ and say that the singular analysis gives the terms $\vec{s}$ and the regular analysis gives the terms $\vec{r}$, then the natural map maps a into $f_i(\vec{r},\vec{s})$. Later in the Herbrand morphisms we will see that in the transformed tree we will have

either $\dfrac{\Gamma_1^* \longrightarrow \Delta_1^*}{\Gamma_1^* \longrightarrow \Delta_1^*}$ or $\dfrac{\Gamma_1^* \longrightarrow F^* fi(\vec{r}, \vec{s}),\Delta_1^*}{\Gamma_1^* \longrightarrow F^* fi(\vec{r}, \vec{s}),\Delta_1^*}$.

So a rule $\longrightarrow \forall$ will go over into a trivial rule.

We are now ready to proceed to the Herbrand morphisms. Our task here is to extend the operations given by the Herbrand processes from sequents to LHK-trees.

<u>Reductions in trees</u> Let $\mathcal{T}$ be a tree over $\Gamma \rightarrow \Delta$, π a position. The tree $\mathrm{RED}_\pi(\mathcal{T})$ is the tree defined by: Say at node in $\mathcal{T}$ we have sequent $\Gamma^* \rightarrow \Delta^*$

i) If no subformula of $\Gamma^* \rightarrow \Delta^*$ properly succeeds position π in $\Gamma \rightarrow \Delta$, then $\Gamma^* \rightarrow \Delta^*$ occurs at node ν in $\mathrm{RED}_\pi(\mathcal{T})$.

ii) Else there are subformulae of $\Gamma^* \rightarrow \Delta^*$ properly succeeding position π in $\Gamma \rightarrow \Delta$. Say the subformulae occur at positions $p_1, \ldots p_N$ in $\Gamma^* \rightarrow \Delta^*$ Then at node ν in $\mathrm{RED}_\pi(\mathcal{T})$ occurs $\mathrm{RED}_{p_1 \ldots p_N}(\Gamma^* \rightarrow \Delta^*)$.

We define the morphism RED from RED_π by applying it for various π until we get an irreducible bottomsequent.

The following theorem is obvious:

THEOREM RED and RED_π are LHK-isomorphisms.

<u>Deletions of places in LHK-trees.</u> Let P be a set of places. The operation DEL_P on LHK-trees is defined by:

Let $\mathcal{T}$ be an LHK-tree over $\Gamma \rightarrow \Delta$. $\mathrm{DEL}_P(\mathcal{T})$ is the tree of sequents got from $\mathcal{T}$ by deleting all formulae in $\mathcal{T}$ with place not in P.

THEOREM If P is a non-empty closed set of places, then DEL_P is an LHK-morphism preserving the bottomsequent.

Proof:

We check on the rules of LHK that: if a formula F of $\mathcal{T}$ has place in P and P is closed, then the immediate predecessor to F has place in P. Besides since P is non-empty and closed, all the places of the formulae in the bottom-sequent of $\mathcal{T}$ are in P. Hence $\mathcal{T}$ and $\mathrm{DEL}_P(\ )$ have the same bottomsequent.

It follows that DEL_P is an LHK-morphism for non-empty and closed P. This is of course not in general true for not closed P.

The auxiliary operations $H_{\pi,\mathfrak{D}}$. π is a position and $\mathfrak{D}$ a finite sequence of terms. Let $\mathcal{T}$ be a tree over $\Gamma \longrightarrow \Delta$. $H_{\pi,\mathfrak{D}}(\mathcal{T})$ is gotten from $\mathcal{T}$ by : Say at node ν in $\mathcal{T}$ there occurs $\Gamma^* \longrightarrow \Delta^*$. Then at node ν in $H_{\pi,\mathfrak{D}}(\mathcal{T})$ we have the sequent gotten from $\Gamma^* \longrightarrow \Delta^*$ by replacing every restricted variable $\forall X$, properly succeeding position π in $\Gamma \longrightarrow \Delta$, by $\underset{X \in \mathfrak{D}}{\mathcal{M}}$-

We do not claim here that $H_{\pi,\mathfrak{D}}$ is an LHK-morphism. In fact it is not so in general - we get into trouble with both $\longrightarrow \forall$ and $\forall \longrightarrow$.

Skolemterms in Herbrand processes. Let $\mathfrak{a}$ be an Herbrand process over $\Gamma \longrightarrow \Delta$. Then $\mathfrak{a}(\Gamma \longrightarrow \Delta)$ is a sequent which is irreducible and without restricted variables. To each general variable $\forall X$ in $(\Gamma \longrightarrow \Delta)$ we assign a Skolemterm $f(\vec{s},\vec{r})$ as follows:

i) f is the Skolemfunction with the index of $\forall X$

ii) $\vec{s}$ are the terms given by the singular analysis of $\forall X$

iii) $\vec{r}$ are the terms given by the regular analysis of $\forall X$.

Since $\mathfrak{a}(\Gamma \longrightarrow \Delta)$ does not contain any restricted variables and is irreducible, then the regular analysis of $\forall X$ gives an analysis of all regular variables binding $\forall X$.

Herbrandprocess acting on LHK-trees. Let $\mathfrak{a}$ be an Herbrand-process over $\Gamma \longrightarrow \Delta$. We define an operation $\mathcal{S}\mathfrak{a}$ on LHK-trees by:

For an LHK-tree $\mathcal{T}$ over $\Gamma \longrightarrow \Delta$, $\mathcal{S}\mathfrak{a}(\mathcal{T})$ is the tree defined by:

i) Put $\mathcal{T}_1 = DEL_P(\mathcal{T})$ where P is the places of $\mathfrak{a}$.

ii) We get $\mathcal{T}_2$ from $\mathcal{T}_1$ by substituting terms for the parameters as given by the narural map of the new parameters in $\mathcal{T}$.

iii) We get $\mathcal{T}_3$ from $\mathcal{T}_2$ by applying RED and $H_{\pi,\mathfrak{D}}$ as given by the Herbrandprocess $\mathfrak{a}$.

iv) $\mathcal{SA}(\mathcal{T})$ is finally gotten from $\mathcal{T}_3$ by for every general variable $\forall X$ in $\mathcal{T}_3$, which properly precedes a general variable in the bottomsequent with assigned Skolemterm T, we substitute T for X in the range of $\forall X$ and delete $\forall X$. (Note that the bottomsequent of $\mathcal{T}_3$ is $\mathcal{A}(\Gamma \longrightarrow \Delta)$ and hence every general variable in the bottomsequent has assigned Skolemterm.)

THEOREM $\mathcal{SA}$ is an LHK-morphism.

Proof:

We must prove that $\mathcal{SA}(\mathcal{T})$ is an LHK-tree.

1. $\mathcal{T}_1$ is an LHK-tree over $\Gamma \longrightarrow \Delta$ since P is a closed non-empty set of places.

2. $\mathcal{T}_2$ may not be a tree. In $\mathcal{T}_2$ we may have

$$\frac{\Gamma_1 \longrightarrow Ff(\vec{s},\vec{r}),\Delta_1}{\Gamma_1 \longrightarrow \forall XFX \quad ,\Delta_1}$$

and hence almost but not quite an instance of $\longrightarrow\forall$. But here $f(\vec{s},\vec{r})$ is the Skolemterm given by the place of $\forall XFX$. - Except for sequences like the above $\mathcal{T}_2$ will be a tree.

3. First observe that in $\mathcal{T}_2$ we can define 'properly succeeding' and hence have no difficulties inapplying RED or $H_{\pi,\mathcal{D}}$ even if $\mathcal{T}_2$ is not a tree.

The operation of RED will not create any difficulty, but it is feasible that the operation $H_{\pi,\mathcal{D}}$ will do. We may have:

$$\frac{\Gamma_2,FT \longrightarrow \Delta_2}{\Gamma_2,\forall XFX \longrightarrow \Delta_2}$$

which goes over into:

$$\frac{\Gamma_2^*, FT \longrightarrow \Delta_2^*}{\Gamma_2^*, \bigwedge FX \longrightarrow \Delta_2^*}$$
$$X \in \mathfrak{D}$$

Here $T \in \mathfrak{D}$ since else FT would have had a place not in P and hence deleted from $\mathcal{T}_1$. We get that each $\forall \longrightarrow$ is either transformed into an application of $\bigwedge \longrightarrow$ or the trivial rule.

4. Our only problem in $\mathcal{T}_3$ are the sequents mentioned in 2. They are taken care of at the last step. Observe that the bottom-sequent in $\mathcal{T}_3$, $\mathcal{A}(\Gamma \longrightarrow \Delta)$, is irreducible without restricted variables. By the rules of LHK and the sequents mentioned in 2 we have only irreducible sequents without restricted variables in $\mathcal{T}_3$. In particular there are no more applications of $\forall \longrightarrow$, $\lambda \longrightarrow$, $\longrightarrow \lambda$.

5. In $\mathcal{T}_3$ every general variable properly precedes a general variable in the bottomsequent. Hence in $\mathfrak{S}\mathcal{A}(\mathcal{T})$ we have no proper variables in the sequents. What happens with the sequent mentioned in 2 ? It is easy to see that $\forall X$ gets assigned Skolemterm $f(\vec{s},\vec{r})$ and hence instead of applications of $\longrightarrow \forall$ we get applications of the trivial rule. We conclude that $\mathfrak{S}\mathcal{A}(\mathcal{T})$ is an LHK-tree without $\forall \longrightarrow$, $\longrightarrow \forall$, $\longrightarrow \lambda$, $\lambda \longrightarrow$.

THEOREM For every Herbrandprocess $\mathcal{A}$, $\mathfrak{S}\mathcal{A}$ is an analyzing morphism.

Proof:

Exercise

LEMMA Let $\mathcal{T}$ be a strong not-secured analyzing LHK-tree over $\Gamma \longrightarrow \Delta$, and $\mathcal{A}$ an Herbrandprocess over $\Gamma \longrightarrow \Delta$. Then $\mathfrak{S}\mathcal{A}(\mathcal{T})$ is not-secured analyzing.

Proof:

$\mathfrak{S}\mathcal{A}(\mathcal{T})$ is analyzing.

Let β be a not secured branch of $\mathcal{T}$.

Assume ν a node of β which is secured in $\mathcal{S}\mathcal{A}(\mathcal{T})$.

Let the sequent at it be $A\vec{s} \longrightarrow A\vec{s}$ where $A\vec{s}$ is an atomic irreducible term.

Since $\mathcal{T}$ is strong, the natural map of the parameters in $\mathcal{T}$ is injective. Hence the terms in $\mathcal{S}\mathcal{A}(\mathcal{T})$ arise from the same parameters in $\mathcal{T}$.

The only way ν is not-secured in $\mathcal{T}$ is that we have either some general variables in the succedent or the formulae may be reducible or both.

For simplicity assume the formulae in $\mathcal{T}$ to be irreducible.

Then at ν in $\mathcal{T}$ we must have

$$A\vec{a}\vec{t} \longrightarrow \forall \vec{X} A\vec{X}\vec{t}$$

but the $\vec{X}$ must be analyzed as $\vec{a}$ since they correspond to the same term in $\mathcal{S}\mathcal{A}(\mathcal{T})$.

Since β is analyzing in $\mathcal{T}$ we must have a node with $A\vec{a}\vec{t} \longrightarrow A\vec{a}\vec{t}$ in β in $\mathcal{T}$.

Contradiction.

Similar considerations work if the formulae at ν in $\mathcal{T}$ are reducible or both reducible and the succeedent contains general variables.

Above there may of course be other formulae at ν than indicated, but they do not affect the argument.

We conclude that $\mathcal{S}\mathcal{A}(\mathcal{T})$ is not-secured.

THEOREM For an Herbrandprocess $\mathcal{A}$, $\mathcal{S}\mathcal{A}$ is a falsifying LHK-morphism.

Proof:

Let $\mathcal{T}$ be an analyzing not secured LHK-tree over $\Gamma \longrightarrow \Delta$.

There is a strong analyzing LHK-tree over $\Gamma \longrightarrow \Delta$, $\mathcal{T}_o$.

By consistency theorem $\mathcal{T}_o$ is not secured.

By lemma $\mathcal{S}\mathfrak{a}(\mathcal{T}_0)$ is not-secured.

By theorem above both $\mathcal{S}\mathfrak{a}(\mathcal{T})$ and $\mathcal{S}\mathfrak{a}(\mathcal{T}_0)$ are analyzing.

By consistency theorem $\mathcal{S}\mathfrak{a}(\mathcal{T})$ is not-secured.

How can we find an $\mathfrak{a}$ which for a secured $\mathcal{T}$ makes $\mathcal{S}\mathfrak{a}(\mathcal{T})$ secured? By the definition of $\mathcal{S}\mathfrak{a}(\mathcal{T})$ we see that the only step which can make a secured mode not- secured is the deletion of formulae with places not among the places of $\mathfrak{a}$. So all we had to do is to let $\mathfrak{a}$ contain enough places.

THEOREM Let $\mathcal{T}$ be a secured LHK-tree over $\Gamma \longrightarrow \Delta$. There is then an Herbrandprocess $\mathfrak{a}$ over $\Gamma \longrightarrow \Delta$ with $\mathcal{S}\mathfrak{a}(\mathcal{T})$ secured.

Proof:

There is a finite set of secured nodes which make $\mathcal{T}$ secured (since $\mathcal{T}$ is finitely branching).

Let P be the set of places of formulae at those nodes. By the last theorem in section IV there is an Herbrand process $\mathfrak{a}$ where P is contained in the places of $\mathfrak{a}$.

It is immediate that in applying $\mathcal{S}\mathfrak{a}(\mathcal{T})$ we let the nodes above remain secured.

Hence $\mathcal{S}\mathfrak{a}(\mathcal{T})$ secured.

THEOREM For two Herbrand processes over $\Gamma \longrightarrow \Delta$, $\mathfrak{a}_1$ and $\mathfrak{a}_2$, and an LHK-tree $\mathcal{T}$ over $\Gamma \longrightarrow \Delta$ with $\mathcal{S}\mathfrak{a}_1(\mathcal{T})$ secured and $\mathfrak{a}_1 < \mathfrak{a}_2$, then $\mathcal{S}\mathfrak{a}_2(\mathcal{T})$ secured.

Proof:

Obvious.

We can of course also formulate Herbrandtheorems for sequents by regarding a sequent as a one-sequent tree. Then $\mathcal{S}\mathfrak{a}(\Gamma \longrightarrow \Delta)$ is defined by substituting Skolemterms for general variables in $\mathfrak{a}(\Gamma \longrightarrow \Delta)$.

We get corollary:

HERBRAND- THEOREM For any sequent $\Gamma \longrightarrow \Delta$:

$\vdash_{LHK} \Gamma \longrightarrow \Delta$ if and only if there is an Herbrandprocess $\mathfrak{a}$

with $\vdash_{LHK} S\mathfrak{a}(\Gamma \longrightarrow \Delta)$.

VI. CONCLUSION

There is one obvious criticism of the above. The notation becomes rather heavy, and to avoid it beeing too heavy I have been a little sloppy, especially in the end of sections IV and V.

The idea should be fairly clear. As mentioned in the introduction I have tried to mix the ideas of my paper [1] with the result of Takahashi [3] and Prawitz [2]. We are forced to introduce indices as names for general variables, and then also distinguish between singular and regular variables binding indices. The Skolemfunctions come out naturally. A difficulty comes in in the formulation of the Herbrand theorem. In first order logic, LK, we can separate the theorem into a Skolemtheorem and an Herbrand theorem. This does not work in higher order logic, LHK. The reason for that is that if we analyze a restricted variable with a λ- term we may generate a new general variable. This is also the reason why, at least superficially, we first get rid of restricted variables and then the general variables in the formulation of the Herbrand theorem in LHK. Until we have gotten rid of all restricted variables we cannot be sure of not getting new general variables. This is not really contrary to LK. If we do not have any λ-terms, it is easy to see that the Herbrand theorem for LHK corresponds to the one for LK.

Our theory for LHK generalizes the one for LK: Does it correspond to Herbrands original idea in any way? It seems to me that the above does full justice to the idea of a finitistic equivalent to the completeness theorem.

VII. REFERENCES

1. JERVELL,HERMAN RUGE: A new proof of the classical Skolem and Herbrand theorems; in preprint from Mathematical Institute, University of Oslo, 1971.

2. PRAWITZ, DAG: Hauptsatz for higher order logic; journal of symbolic logic XXXIII, 1968.

3. TAKAHASHI,MOTO-O: A proof of the cut-elimination theorem in simple type- theory; journal of the Mathematical Society of Japan, vol.19,1967.